

ARC-TRANSITIVE PENTAVALENT CAYLEY GRAPHS WITH SOLUBLE VERTEX STABILIZER ON FINITE NONABELIAN SIMPLE GROUPS*

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ABSTRACT. A Cayley graph $\Gamma = \text{Cay}(G, S)$ is said to be normal if G is normal in $\text{Aut}\Gamma$. The concept of normal Cayley graphs was first proposed by M.Y.Xu in [Discrete Math. 182, 309-319, 1998] and it plays an important role in determining the full automorphism groups of Cayley graphs. In this paper, we investigate the normality problem of the connected arc-transitive pentavalent Cayley graphs with soluble vertex stabilizer on finite nonabelian simple groups. We prove that all such graphs Γ are either normal or $G = A_{39}$ or A_{79} . Further, a connected arc-transitive pentavalent Cayley graph on A_{79} is constructed. To our knowledge, this is the first known example of pentavalent 3-arc-transitive Cayley graph on finite nonabelian simple group which is non-normal.

KEYWORDS. Simple group; Normal Cayley graph; Arc-transitive graph

1. INTRODUCTION

All graphs are assumed to be finite, simple and undirected.

Let Γ be a graph. We use $V\Gamma$, $E\Gamma$ and $\text{Aut}\Gamma$ to denote the vertex set, edge set and automorphism group of Γ , respectively. Denote $\text{val}\Gamma$ the valency of Γ . Let $X \leq \text{Aut}\Gamma$ and let s be a positive integer. The graph Γ is said to be (X, s) -arc-transitive, if X acts transitively on the set of s -arcs of Γ , where an s -arc is an $(s + 1)$ -tuple (v_0, v_1, \dots, v_s) of $s + 1$ vertices satisfying $(v_{i-1}, v_i) \in E\Gamma$ and $v_{i-1} \neq v_{i+1}$ for all i . The graph Γ is called (X, s) -transitive if it is (X, s) -arc-transitive but not $(X, s + 1)$ -arc-transitive. In particular, an $(\text{Aut}\Gamma, s)$ -arc-transitive or $(\text{Aut}\Gamma, s)$ -transitive graph is just called s -arc-transitive or s -transitive graph; and 0-arc-transitive graph is called *vertex transitive* graph, 1-arc-transitive graph is called *arc-transitive* graph or *symmetric* graph.

Let G be a finite group with identity 1, and let S be a subset of G such that $1 \notin S$ and $S = S^{-1} := \{x^{-1} \mid x \in S\}$. The Cayley graph of G with respect to S , denoted by $\text{Cay}(G, S)$, is defined on G such that $g, h \in G$ are adjacent if and only if $hg^{-1} \in S$. Then $\text{Cay}(G, S)$ is a regular graph of valency $|S|$. It is well-known that Γ is connected if and only if $\langle S \rangle = G$, that is, S is a generating set of the group G . For a Cayley graph $\text{Cay}(G, S)$, the underlying group G can be viewed as a regular subgroup of $\text{AutCay}(G, S)$ which acts on G by right multiplication.

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Conversely, a graph Γ is isomorphic to a Cayley graph if and only if $\text{Aut}\Gamma$ has a regular subgroup, refer to [1, Proposition 16.3].

Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph. Set

$$\text{Aut}(G, S) = \{\sigma \in \text{Aut}(G) \mid S^\sigma = S\}.$$

Then $\text{Aut}(G, S)$, acting on G naturally, is a subgroup of $\text{Aut}\Gamma$. If Γ is connected, then $\text{Aut}(G, S)$ acts faithfully on S and lies in the stabilizer of the vertex corresponding to the identity of G . Moreover, the normalizer $N_{\text{Aut}\Gamma}(G)$ equals to the semi-directed product $G:\text{Aut}(G, S)$, see [8].

A Cayley graph $\Gamma = \text{Cay}(G, S)$ is said to be *normal* if G is normal in $\text{Aut}\Gamma$, that is, $\text{Aut}\Gamma = G:\text{Aut}(G, S)$, refer to [17]; otherwise, Γ is called *non-normal*. Thus, for a connected normal Cayley graph $\Gamma = \text{Cay}(G, S)$, the group $\text{Aut}(G, S)$ is just the stabilizer in $\text{Aut}\Gamma$ of the vertex corresponding to the identity of G .

In this paper we consider connected arc-transitive pentavalent Cayley graphs.

The concept of normal Cayley graphs was first proposed by M.Y.Xu in [17] and it plays an important role in determining the full automorphism groups of Cayley graphs. The Cayley graphs on finite nonabelian simple groups are received most attention in the literature. For example, X.G.Fang, C.E.Praeger and J.Wang [5] gave a general description of the possibilities for the automorphism groups of connected Cayley graphs on a finite non-abelian simple group. Then some further work focuses on the small valencies because the precise structure of the vertex stabilizer of arc-transitive cubic, tetravalent and pentavalent graphs was determined by a series of papers. Let $\Gamma = \text{Cay}(G, S)$ be a connected arc-transitive Cayley graph on a finite nonabelian simple group G . For the cubic case, C.H.Li [10] proved that only 7 groups are exceptions for Γ being not normal; on the basis of C.H.Li's result, S.J.Xu et al. [18, 19] proved that all such Γ are normal except two 5-transitive Cayley graphs of the alternating group A_{47} , and so a complete classification of cubic s -transitive non-normal Cayley graphs of finite simple groups was given. For the tetravalent case, X.G.Fang, C.H.Li and M.Y.Xu in [4] proved that most of such graphs are normal except a list of possible G . Recently, J.J.Li, J.C.Ma and the first author of the present paper in [12] proved that all s -regular Γ (that is, $\text{Aut}\Gamma$ acts regularly on its s -arc set) are normal except the case $G = A_{35}$, and a 3-arc-transitive non-normal Cayley graph on A_{35} was constructed. Further, X.G.Fang, J.Wang and S.M.Zhou in [6] proved that all 2-transitive Γ are normal except two graphs on M_{11} . For the pentavalent case, J.X.Zhou and Y.Q.Feng in [20] proved that all 1-transitive Γ are normal, and in [13] the authors of the present paper constructed a 2-arc-transitive non-normal Cayley graph on A_{39} . More results about normality of Cayley graphs we refer the reader to a survey paper in [7].

Examples of connected arc-transitive non-normal cubic, tetravalent and pentavalent Cayley graphs on nonabelian simple groups are very rare (the known examples are only the above mentioned graphs on A_{47} , M_{11} , A_{35} and A_{39}), we concentrate on the pentavalent case in this paper. In particular, we construct a connected 3-arc-transitive non-normal pentavalent Cayley graph on A_{79} in Construction 4.1. It is shown in [11] that 3-arc-transitive Cayley graphs of any given valency are rare.

The aim of this paper is to investigate the normality problem of the connected arc-transitive pentavalent Cayley graphs with soluble vertex stabilizer on finite nonabelian simple groups. Our main result is the following theorem.

Theorem 1.1. *Let G be a finite nonabelian simple group, and let $\Gamma = \text{Cay}(G, S)$ be an arc-transitive pentavalent Cayley graph on G . Then the following statements hold.*

- (1) *Either Γ is a normal Cayley graph or $G = A_{39}$ or A_{79} . Further,*
- (2) *there exist connected arc-transitive pentavalent non-normal Cayley graphs for $(G, \text{Aut}\Gamma) = (A_{39}, A_{40})$ or (A_{79}, A_{80}) .*

Remark 1.1. (a) The connected 2-arc-transitive non-normal pentavalent Cayley graph on A_{39} in part (2) was constructed by the authors of the present paper in [13, Construction 3.1].

(b) The connected 3-arc-transitive non-normal pentavalent Cayley graph on A_{79} in part (2) constructed in Section 4 is the first known example in 3-arc-transitive case.

2. PRELIMINARIES

We give some necessary preliminary results in this section. The first one is a property of the Fitting subgroup, see [16, P. 30, Corollary].

Lemma 2.1. *Let F be the Fitting subgroup of a group G . If G is soluble, then $F \neq 1$ and the centralizer $C_G(F) \leq F$.*

The next lemma is about primitive permutation groups of degree less than 80, refer to [15].

Lemma 2.2. *Let T be a primitive permutation group on Ω and let K be the stabilizer of a point $w \in \Omega$. If T is a nonabelian simple group and $|\Omega|$ divides 80, then $(T, K, |\Omega|)$ is one of the following Table 1.*

T	A_8	A_{10}	A_{16}	A_{20}	$\text{PSL}(4, 3)$	A_{40}	A_{80}
K	A_7	A_9	A_{15}	A_{19}	$\mathbb{Z}_3^3 : \text{PSL}(3, 3)$	A_{39}	A_{79}
$ \Omega $	8	10	16	20	40	40	80

TABLE 1. Primitive permutation groups of degree less than 80

Simple groups which have subgroups of index dividing $2^5 \cdot 3^2$ are given in the following lemma, refer to [4, Lemma 2.4].

Lemma 2.3. *Let T be a non-abelian simple group which has a subgroup L of index dividing $2^5 \cdot 3^2$. Then T , L and $n := |T : L|$ are given in the following Table 2.*

We next introduce the definition of coset graph. Let G be a finite group and let H be a core-free subgroup of G . Define the *coset graph* $\text{Cos}(G, H, g)$ of G with

T	L	n	Remark
A_n	A_{n-1}	n	$n \mid 2^5 \cdot 3^2$
M_{11}	$\text{PSL}(2, 11)$	12	
M_{12}	M_{11}	12	
M_{24}	M_{23}	24	

TABLE 2. Simple groups with having subgroups of index dividing $2^5 \cdot 3^2$

respect to H as the graph with vertex set $[G : H]$ such that Hx, Hy are adjacent if and only if $yx^{-1} \in HgH$. The following lemma about coset graphs are well known and the proof of the lemma follows from the definition of coset graphs.

Lemma 2.4. *Using notation as above. Let $\text{val}\Gamma$ be the valency of Γ . Then the coset graph $\Gamma = \text{Cos}(G, H, g)$ is G -arc-transitive graph and*

- (1) $\text{val}\Gamma = |H : H \cap H^g|$;
- (2) Γ is connected if and only if $\langle H, g \rangle = G$.
- (3) If G has a subgroup R acting regularly on the vertices of $\text{Cos}(G, H, g)$, then $\text{Cos}(G, H, g) \cong \text{Cay}(R, S)$, where $S = R \cap HgH$.

Conversely, each G -arc-transitive graph Σ is isomorphic to the coset graph $\text{Cos}(G, G_v, g)$, where $g \in N_G(G_{vw})$ is a 2-element such that $g^2 \in G_v$, and $v \in V\Sigma$, $w \in \Sigma(v)$.

For a graph Γ and a vertex-transitive subgroup $X \leq \text{Aut}\Gamma$. Let N be an intransitive normal subgroup of X on $V\Gamma$. Denote V_N the set of N -orbits in $V\Gamma$. The *normal quotient graph* Γ_N defined as the graph with vertex set V_N and two N -orbits $B, C \in V_N$ are adjacent in Γ_N if and only if some vertex of B is adjacent in Γ to some vertex of C . By [14, Theorem 9], we have the following lemma.

Lemma 2.5. *Let Γ be an arc-transitive graph of prime valency $p > 2$ and let X be an arc-transitive subgroup of $\text{Aut}\Gamma$. If a normal subgroup N of X has more than two orbits on $V\Gamma$, then Γ_N is an X/N -arc-transitive graph of valency p and N is semiregular on $V\Gamma$.*

The following lemma is about the stabilizers of arc-transitive pentavalent graphs, refer to [9, 20].

Lemma 2.6. *Let Γ be a pentavalent (G, s) -transitive graph, where $G \leq \text{Aut}\Gamma$ and $s \geq 1$. Let $\alpha \in V\Gamma$. Then one of the following holds, where D_{10} , D_{20} and F_{20} denote the dihedral groups of order 10, 20, and the Frobenius group of order 20, respectively.*

- (a) *If G_α is soluble, then $s \leq 3$ and $|G_\alpha| \mid 80$. Further, the couple (s, G_α) lies in the following Table 3.*
- (b) *If G_α is insoluble, then $2 \leq s \leq 5$, and $|G_\alpha| \mid 2^9 \cdot 3^2 \cdot 5$. Further, the couple (s, G_α) lies in the following Table 4.*

s	1	2	3
G_α	$\mathbb{Z}_5, D_{10}, D_{20}$	$F_{20}, F_{20} \times \mathbb{Z}_2$	$F_{20} \times \mathbb{Z}_4$

TABLE 3. The soluble case

s	2	3	4	5
G_α	A_5, S_5	$A_4 \times A_5, (A_4 \times A_5) : \mathbb{Z}_2, S_4 \times S_5$	$ASL(2, 4), AGL(2, 4), A\Sigma L(2, 4), A\Gamma L(2, 4)$	$\mathbb{Z}_2^6 : \Gamma L(2, 4)$
$ G_\alpha $	60, 120	720, 1440, 2880	960, 1920, 2880, 5760	23040

TABLE 4. The insoluble case

3. THE PROOF OF NORMAL CASE

Let $\Gamma := \text{Cay}(G, S)$ be an arc-transitive pentavalent Cayley graph, where G is a finite nonabelian simple group. Let $A := \text{Aut}\Gamma$ and A_v be the stabilizer of v in A where $v \in V\Gamma$. Assume that A_v is soluble. Then by Lemma 2.6, $|A_v|$ divides 80.

The following lemma consider the case A has no nontrivial soluble normal subgroup.

Lemma 3.1. *Assume that A has no nontrivial soluble normal subgroup. Then G is either normal in A or $G = A_{39}$ or A_{79} .*

Proof. Let N be a minimal normal subgroup of A . Then $N = T^d$, where $d \geq 1$ and T is a nonabelian simple group.

Assume that G is not normal in A . Then since $N \cap G \trianglelefteq G$ and G is a nonabelian simple group, $N \cap G = 1$ or G . If $N \cap G = 1$ then since $A = GA_v$, $|N| \mid |A_v| \mid 80$, which is a contradiction since N is insoluble. Hence $N \cap G = G$, $G \leq N$. If $G = N$, then $G \trianglelefteq A$, a contradiction to the assumption. Thus $G < N$. Assume that $d \geq 2$. Then $N = T_1 \times T_2 \times \dots \times T_d$ where $d \geq 2$ and $T_i \cong T$ is a nonabelian simple group. Since $T_1 \cap G \trianglelefteq G$, we have $T_1 \cap G = 1$ or G . If $T_1 \cap G = 1$, then $|T_1| \mid |A_v| \mid 80$, a contradiction. If $T_1 \cap G = G$, then $G \leq T_1$. It follows that $|T_2| \mid |A_v| \mid 80$, which is also a contradiction. Thus, $d = 1$ and $N = T$ is a nonabelian simple group. Then $T = GT_v$, $T_v \neq 1$ and $|T_v|$ divides 80. Since T has the proper subgroup G with index dividing 80, we can take a maximal proper subgroup K of T which contains G as a subgroup. Let $\Omega = [T : K]$. Then $|\Omega|$ divides 80 and T has a primitive permutation representation on Ω , of degree $n := |\Omega|$. Since T is simple, this representation is faithful and thus T is a primitive permutation group of degree n . Note that K is the stabilizer of a point $w \in \Omega$, that is, $K = T_w$. Since T is nonabelian simple, $n > 4$. Consequently, by Lemma 2.2, we have $(T, K, |\Omega|)$ is listed in Table 1.

Assume that $(T, K, |\Omega|) = (\text{PSL}(4, 3), \mathbb{Z}_3^3 : \text{PSL}(3, 3), 40)$. Then since $G \leq K$ and G is a nonabelian simple group, we have that G is a proper subgroup of K . Since $|T : G| \mid 80$ and $|\Omega| = 40$, we have $|K : G| = 2$. It is easy to see that a subgroup of K with index 2 can not be a nonabelian simple group, which is a contradiction.

Assume that $(T, K, |\Omega|) = (A_8, A_7, 8)$ or $(A_{16}, A_{15}, 16)$. Then since $G \leq K$, G is a nonabelian simple group and $|T : G| \leq 80$, we have $G = K$. Since Γ is connected, $T \trianglelefteq A$ and $T_v \neq 1$, we have $1 \neq T_v^{\Gamma(v)} \trianglelefteq A_v^{\Gamma(v)}$. Since Γ is A -arc-transitive of valency 5, it follows that $A_v^{\Gamma(v)}$ is primitive on $\Gamma(v)$ and so $T_v^{\Gamma(v)}$ is transitive on $\Gamma(v)$, $5 \mid |T_v| = |T : G| = |\Omega|$, a contradiction. Finally, by [20, Theorem 5.4], $(T, K, |\Omega|) \neq (A_{20}, A_{19}, 20)$ or $(A_{10}, A_9, 10)$. Thus, we have $G = A_{39}$ or A_{79} , the lemma holds. \blacksquare

The following lemma consider the case A has a nontrivial soluble normal subgroup.

Lemma 3.2. *Assume that A has a nontrivial soluble normal subgroup. Then G is either normal in A or $G = A_{39}$.*

Proof. Let M be the largest soluble normal subgroup of A . Then $M \text{ char } A$. Since A has a nontrivial soluble normal subgroup, we have $M \neq 1$. Since $M \cap G \trianglelefteq G$ and G is simple, we have $M \cap G = 1$ and so $|M| \mid |A_v| \mid 80$. Since $|VG| = |G|$ contains at least three prime factors, it follows that M has more than two orbits on VG . By Lemma 2.5, M is semi-regular on VG .

Let $\bar{A} = A/M$ and let $\bar{\Gamma} = \Gamma_M$. Then by Lemma 2.5, $\bar{\Gamma}$ is \bar{A} -arc-transitive. Let \bar{N} be a minimal normal subgroup of \bar{A} and let N be the full preimage of \bar{N} under $A \rightarrow A/M$. By the maximality of M , \bar{N} is insoluble. Then $\bar{N} = T_1 \times T_2 \times \dots \times T_d = T^d$, where T is a nonabelian simple group and $d \geq 1$.

We first show that $d = 1$. Let $\bar{G} = GM/M$. Then $\bar{G} \cong G$ is a nonabelian simple group. Since $\bar{N} \cap \bar{G} \trianglelefteq \bar{G}$, we have $\bar{N} \cap \bar{G} = 1$ or \bar{G} . If $\bar{N} \cap \bar{G} = 1$, then $|\bar{N}|$ divides 80, which is a contradiction since \bar{N} is insoluble. Hence $\bar{G} \leq \bar{N}$. Since \bar{G} is simple, $|\bar{G}|$ must divide the order of some composition factor of \bar{N} , that is, $|\bar{G}| \mid |T_1|$. If $d \geq 2$ then $|T_2|$ divides $|\bar{N} : \bar{G}|$ which divides $|\bar{A}_{\bar{v}}|$, which is not possible since $\bar{A}_{\bar{v}}$ is a $\{2, 5\}$ -group and T_2 is simple, where $\bar{v} \in V\bar{\Gamma}$. Therefore, $d = 1$ and \bar{N} is a nonabelian simple group. This argument also proves that \bar{N} is the unique insoluble minimal normal subgroup of \bar{A} . Thus $\bar{N} \text{ char } A$ and $N \text{ char } A$.

Assume first that $\bar{G} = \bar{N}$. Then $N = M : G$. If G centralizes M then $N = M \times G$, and therefore $G \text{ char } N \text{ char } A$, a contradiction. Thus G does not centralize M . It follows that $\text{Aut}(M)$ is insoluble.

Let F be the Fitting subgroup of M . By Lemma 2.1, $F \neq 1$ and $C_M(F) \leq F$. Since $|M|$ divides 80, we have $F = O_2(M) \times O_5(M)$, where $O_2(M)$, $O_5(M)$ denote the largest normal 2-, 5-subgroups of M , respectively. Clearly $|O_2(M)| \mid 16$ and $|O_5(M)| \mid 5$. Assume that $|O_2(M)| \leq 2$. Then F is abelian and $F = C_M(F)$. Since $M/C_M(F) \lesssim \text{Aut}(F)$, we have $M \lesssim F \cdot \text{Aut}(F)$. If $|O_2(M)| = 1$, then since $O_5(M) \leq \mathbb{Z}_5$, we have $F \leq \mathbb{Z}_5$. It follows that $M \lesssim \mathbb{Z}_5 \cdot \mathbb{Z}_4$, and so $\text{Aut}(M)$ is soluble, a contradiction. If $|O_2(M)| = 2$, then $F \leq \mathbb{Z}_{10}$. Thus, $M \lesssim \mathbb{Z}_{10} \cdot \mathbb{Z}_4$. A computation by Magma [2], $\text{Aut}(M)$ is soluble, a contradiction. Hence $|O_2(M)| \geq 4$.

Let $R = O_2(M)$. Then $R \text{ char } M \text{ char } A$. Let $B = RG$. We claim that B is not normal in A . Suppose to the contrary that B is normal in A . Then $B_v^{\Gamma(v)} \trianglelefteq A_v^{\Gamma(v)}$.

Note that $A_v^{\Gamma(v)}$ is primitive on $\Gamma(v)$. Since $B > G$, $B_v \neq 1$ and therefore $B_v^{\Gamma(v)} \neq 1$ is transitive on $\Gamma(v)$. Thus 5 divides $|B_v| = |B : G| = |R|$, a contradiction. So B is not normal in A as claimed.

Assume that G does not centralize R . Since $R \trianglelefteq B$, we have that $B/C_B(R)$ is isomorphic to a subgroup of $\text{Aut}(R)$. Since G is nonabelian simple, it follows that $|R| \geq 8$ and so $|M/R| \leq 80/8 = 10$. Therefore $\text{Aut}(M/R)$ is soluble. Since $N/R = (M/R) : (RG/R)$ and $RG/R \cong G$ is nonabelian simple, we have RG/R centralizes M/R , and so $N/R = (M/R) \times (RG/R)$. It follows that $RG/R \text{ char } N/R$, $RG \text{ char } N \trianglelefteq A$, which is a contradiction to the conclusion in the previous paragraph.

Thus G centralizes R . Since G does not centralize M , $R \neq M$. Since $|R| \geq 4$, it follows that $|M/R| \mid 20$, and so $\text{Aut}(M/R)$ is soluble. Similar arguments to the previous paragraph lead to $RG \trianglelefteq A$, a contradiction.

Thus $\bar{G} \neq \bar{N}$, \bar{G} is a proper subgroup of \bar{N} and $|\bar{N} : \bar{G}|$ divides 40. Let \bar{K} be a maximal proper subgroup of \bar{N} which contains \bar{G} as a subgroup and let $\bar{\Omega} = [\bar{N} : \bar{K}]$. Then by Lemma 2.2, we have $(\bar{N}, \bar{K}, |\bar{\Omega}|)$ is listed in Table 1. Since $|\bar{\Omega}|$ divides 40, we have $(\bar{N}, \bar{K}, |\bar{\Omega}|) \neq (A_{80}, A_{79}, 80)$. If $(\bar{N}, \bar{K}, |\bar{\Omega}|) = (\text{PSL}(4, 3), \mathbb{Z}_3^3 : \text{PSL}(3, 3), 40)$, then since $|\bar{N} : \bar{G}| \mid 40$ and $|\bar{\Omega}| = 40$, we have $\bar{G} = \bar{K}$, which is a contradiction since \bar{G} is a nonabelian simple group. Further, by the proof in [20, Theorem 5.4], $(\bar{N}, \bar{K}, |\bar{\Omega}|) \neq (A_{20}, A_{19}, 20)$ or $(A_{10}, A_9, 10)$. Thus, to complete the proof of the lemma, we only need to exclude the cases where $(\bar{N}, \bar{K}, |\bar{\Omega}|) = (A_{16}, A_{15}, 16)$ or $(A_8, A_7, 8)$.

Suppose that $(\bar{N}, \bar{K}, |\bar{\Omega}|) = (A_{16}, A_{15}, 16)$ or $(A_8, A_7, 8)$. We claim that there exists $L \trianglelefteq N$ such that 5 divides $|N : L|$. Let \bar{v} be a vertex in $V\bar{\Gamma}$. Then $|\bar{N}_{\bar{v}} : \bar{G}_{\bar{v}}| = |\bar{N} : \bar{G}| = 16$ or 8. Since $\bar{\Gamma}$ is a pentavalent \bar{A} -arc-transitive graph, we have $\bar{A}_{\bar{v}}^{\bar{\Gamma}(\bar{v})}$ is primitive on $\bar{\Gamma}(\bar{v})$. Since $1 \neq \bar{N}_{\bar{v}}^{\bar{\Gamma}(\bar{v})} \trianglelefteq \bar{A}_{\bar{v}}^{\bar{\Gamma}(\bar{v})}$, $\bar{N}_{\bar{v}}^{\bar{\Gamma}(\bar{v})}$ is transitive on $\bar{\Gamma}(\bar{v})$ and so 5 divides $|\bar{N}_{\bar{v}}|$. Therefore, 5 divides $|\bar{G}_{\bar{v}}|$. Since G is regular on $V\Gamma$, it follows that $|M| = |\bar{G}_{\bar{v}}|$. If $(\bar{N}, \bar{G}, |\bar{\Omega}|) = (A_{16}, A_{15}, 16)$, then $|M| = 5$. Consequently, $N \cong \mathbb{Z}_5.A_{16}$, an extension of \mathbb{Z}_5 by A_{16} . By Atlas [3], the Schur multiplier of A_{16} equals \mathbb{Z}_2 , and therefore, $N \cong \mathbb{Z}_5 \times A_{16}$. So there exists $L \cong A_{16}$ as claimed. If $(\bar{N}, \bar{G}, |\bar{\Omega}|) = (A_8, A_7, 8)$, then $|M| = 5$ or 10. If $|M| = 5$, then arguing as for the case $\bar{N} = A_{16}$, there exists $L \cong A_8$ as claimed. Thus we suppose that $|M| = 10$. Then $M \cong \mathbb{Z}_{10}$ or D_{10} , and we can conclude that $N \cong M_5 \times (\mathbb{Z}_2.A_8)$ or $(M_5 \times A_8).\mathbb{Z}_2$, where $M_5 \cong \mathbb{Z}_5$ is a Sylow 5-subgroup of M . Thus $L \cong \mathbb{Z}_2.A_8$ or A_8 exists as claimed. Since $|L| \neq |G|$, L is not regular on $V\Gamma$. Now $1 \neq L_v \trianglelefteq N_v$, and since N_v is transitive (and so primitive) on $\Gamma(v)$, L_v is transitive on $\Gamma(v)$ and so 5 divides $|L_v|$. Consequently, 5^2 divides $|N : L| \cdot |L : G| = |N : G| = |N_v|$, which is a contradiction to Lemma 2.6. Therefore, $G \cong \bar{G} \not\cong A_{15}$ or A_7 , and so $G \cong A_{39}$. This completes the proof of the lemma. ■

4. A 3-ARC-TRANSITIVE NON-NORMAL PENTAVALENT CAYLEY GRAPH ON A_{79}

By [13, Theorem 1.1], there exists a connected 2-arc-transitive non-normal pentavalent Cayley graph on A_{39} with full automorphism group A_{40} . In this section, we

will construct a connected 3-arc-transitive non-normal pentavalent Cayley graph on A_{79} and prove its full automorphism group isomorphic to A_{80} .

Construction 4.1. Let $G := \text{Alt}(\{2, 3, \dots, 80\}) = A_{79}$ and let $H = \langle a, b, c \rangle < X := \text{Alt}(\{1, 2, \dots, 80\}) = A_{80}$, where

$$\begin{aligned} a = & (1\ 16\ 11\ 6)(2\ 17\ 12\ 7)(3\ 18\ 13\ 8)(4\ 19\ 14\ 9)(5\ 20\ 15\ 10) \\ & (21\ 36\ 31\ 26)(22\ 37\ 32\ 27)(23\ 38\ 33\ 28)(24\ 39\ 34\ 29)(25\ 40\ 35\ 30) \\ & (41\ 56\ 51\ 46)(42\ 57\ 52\ 47)(43\ 58\ 53\ 48)(44\ 59\ 54\ 49)(45\ 60\ 55\ 50) \\ & (61\ 76\ 71\ 66)(62\ 77\ 72\ 67)(63\ 78\ 73\ 68)(64\ 79\ 74\ 69)(65\ 80\ 75\ 70), \\ b = & (1\ 46\ 77\ 35)(2\ 43\ 66\ 28)(3\ 60\ 75\ 21)(4\ 57\ 64\ 34)(5\ 54\ 73\ 27) \\ & (6\ 51\ 62\ 40)(7\ 48\ 71\ 33)(8\ 45\ 80\ 26)(9\ 42\ 69\ 39)(10\ 59\ 78\ 32) \\ & (11\ 56\ 67\ 25)(12\ 53\ 76\ 38)(13\ 50\ 65\ 31)(14\ 47\ 74\ 24)(15\ 44\ 63\ 37) \\ & (16\ 41\ 72\ 30)(17\ 58\ 61\ 23)(18\ 55\ 70\ 36)(19\ 52\ 79\ 29)(20\ 49\ 68\ 22), \\ c = & (1\ 17\ 13\ 9\ 5)(2\ 18\ 14\ 10\ 6)(3\ 19\ 15\ 11\ 7)(4\ 20\ 16\ 12\ 8) \\ & (21\ 37\ 33\ 29\ 25)(22\ 38\ 34\ 30\ 26)(23\ 39\ 35\ 31\ 27)(24\ 40\ 36\ 32\ 28) \\ & (41\ 57\ 53\ 49\ 45)(42\ 58\ 54\ 50\ 46)(43\ 59\ 55\ 51\ 47)(44\ 60\ 56\ 52\ 48) \\ & (61\ 77\ 73\ 69\ 65)(62\ 78\ 74\ 70\ 66)(63\ 79\ 75\ 71\ 67)(64\ 80\ 76\ 72\ 68). \end{aligned}$$

Take $x_1 \in G$ as follows:

$$\begin{aligned} x_1 = & (2\ 22)(3\ 29)(4\ 36)(5\ 23)(6\ 35)(7\ 68)(8\ 79)(9\ 70)(10\ 61)(11\ 77) \\ & (12\ 49)(13\ 52)(14\ 55)(15\ 58)(16\ 46)(17\ 20)(18\ 19)(21\ 34)(24\ 60) \\ & (25\ 62)(26\ 64)(27\ 28)(30\ 51)(31\ 57)(32\ 66)(33\ 73)(37\ 43)(38\ 54) \\ & (39\ 75)(42\ 65)(44\ 53)(45\ 74)(47\ 50)(48\ 63)(56\ 72)(59\ 76)(69\ 80)(71\ 78). \end{aligned}$$

Define $\Sigma = \text{Cos}(X, H, x_1)$.

Lemma 4.1. The graph $\Sigma = \text{Cos}(X, H, x_1)$ in Construction 4.1 is connected, 3-arc-transitive and isomorphic to the non-normal Cayley graph $\text{Cay}(G, S)$ of G , determined by $S = \{x_1, x_2, x_2^{-1}, x_3, x_3^{-1}\}$ with

$$\begin{aligned} x_2 = & (2\ 5\ 40\ 79\ 75\ 49\ 67\ 36\ 10\ 74\ 37\ 8\ 72\ 62\ 14\ 56\ 18\ 4\ 33\ 70\ 64\ 52\ 34 \\ & 77\ 43\ 69\ 11\ 65\ 30\ 7\ 39\ 58\ 35\ 25\ 44\ 21\ 24\ 9\ 63\ 29\ 31\ 55\ 22\ 38\ 47\ 20) \\ & (3\ 26\ 66\ 46\ 78\ 15\ 59\ 19)(6\ 27\ 68\ 71\ 13\ 53\ 17\ 50\ 16\ 42\ 45\ 57\ 76\ 23) \\ & (12\ 61\ 73\ 60\ 80)(28\ 32\ 41)(48\ 54), \\ x_3 = & (2\ 20)(3\ 4\ 25\ 8\ 38\ 7\ 26\ 43\ 44\ 68\ 35\ 76\ 48\ 19)(5\ 32\ 54\ 16\ 58\ 75\ 36\ 24) \\ & (6\ 39\ 46\ 56\ 21\ 78\ 42\ 27\ 65\ 51\ 15\ 55\ 47\ 77\ 22\ 23\ 37\ 9\ 71\ 63\ 13\ 80\ 74 \\ & 59\ 34\ 69\ 70\ 10\ 62\ 72\ 12\ 64\ 28\ 67\ 45\ 40\ 57\ 17\ 41\ 79\ 14\ 52\ 50\ 66\ 29\ 31) \\ & (11\ 73\ 61)(18\ 49\ 53\ 33\ 60). \end{aligned}$$

Proof. Let $\Delta = \{1, 2, \dots, 80\}$ and consider the natural action of X on Δ . By Magma [2], $\langle H, x_1 \rangle = X$, and so Σ is connected by Lemma 2.4 (2). Since $c^b = c^2$ and a centralizes $\langle b, c \rangle$, it follows that $H = \langle a, b, c \rangle = \langle b, c \rangle \times \langle a \rangle \cong (\mathbb{Z}_5 : \mathbb{Z}_4) \times \mathbb{Z}_4$. Furthermore, it is easy to see that H is transitive on Δ and so is regular on Δ . Hence X has a factorization $X = GH = HG$ with $G \cap H = 1$. Therefore, Σ is isomorphic to a Cayley graph of $G = A_{79}$. Further computation shows that $\frac{|H|}{|H \cap H^{x_1}|} = 5$. By Lemma 2.4 (1) we have that Σ is pentavalent. Since $H \cong (\mathbb{Z}_5 : \mathbb{Z}_4) \times \mathbb{Z}_4$, we have Σ is 3-arc-transitive by Lemma 2.6. Since X is simple, G is not normal in $X \leq \text{Aut} \Sigma$. Thus Σ is non-normal. Let x_2, x_3 and S define as in

this lemma. Computation shows that $G \cap (Hx_1H) = S$. Then $\Sigma \cong \text{Cay}(G, S)$ by Lemma 2.4 (3). This completes the proof of the lemma. ■

In the next lemma we show that $\text{Aut}\Sigma \cong A_{80}$. This will therefore complete the proof of Theorem 1.1.

Lemma 4.2. *The full automorphism group $\text{Aut}\Sigma$ of the graph $\Sigma = \text{Cos}(X, H, x_1)$ in Construction 4.1 is isomorphic to A_{80} .*

Proof. Let $A = \text{Aut}\Sigma$. Assume first that A is quasiprimitive on $V\Sigma$. Let N be a minimal normal subgroup of A . Then N acts transitively on $V\Sigma$, and so N is insoluble. Then $N = T^d$ with T a nonabelian simple group and $d \geq 1$. Let p be the largest prime factor of $|A_{79}|$. Then $p > 5$ and $p^2 \nmid |A_{79}|$. Since N is transitive on $V\Sigma$ and $|V\Sigma| = |A_{79}|$, we have $p \mid |N|$. Suppose that $d \geq 2$. Then $p^d \mid |N|$. However, by Lemma 2.6, $|A_v| \mid 2^9 \cdot 3^2 \cdot 5$, and so $p^d \mid |N| \mid |A| \mid 2^9 \cdot 3^2 \cdot 5 \cdot |A_{79}|$, a contradiction. Hence $d = 1$ and $N = T \trianglelefteq A$. Let $C = C_A(T)$. Then $C \trianglelefteq A$ and $CT = C \times T$. If $C \neq 1$, then C is transitive on $V\Sigma$ as A is quasiprimitive on $V\Sigma$. It follows that $p \mid |C|$. Therefore, $p^2 \mid |CT| \mid |A|$, a contradiction. Hence $C = 1$ and $A \leq \text{Aut}(T)$ is almost simple.

Since $T \cap X \trianglelefteq X \cong A_{80}$, it follows that $T \cap X = 1$ or A_{80} . If $T \cap X = 1$, then $|T| \mid \frac{|A|}{|X|} \mid 2^5 \cdot 3^2$, and so T is soluble, a contradiction. Thus $T \cap X = X$, and so $X \leq T$. It follows that $|T : X| \mid |A : X| \mid 2^5 \cdot 3^2$. By Lemma 2.3 we can conclude that $T = X \cong A_{80}$. Thus $A \leq \text{Aut}(T) \cong S_{80}$. If $A \cong S_{80}$, then $|A_v| = \frac{|A|}{|G|} = 160$, a contradiction with Lemma 2.6. Hence $A \cong A_{80}$.

Now assume that A is not quasiprimitive on $V\Sigma$. Let M be a minimal normal subgroup of A which is not transitive on $V\Sigma$. Then $M \cap X \trianglelefteq X$. It follows that $M \cap X = 1$ or A_{80} . If $M \cap X = A_{80}$, then $X \leq M$, and so M is transitive on $V\Sigma$, a contradiction. If $M \cap X = 1$, then $|M|$ divides $\frac{|A|}{|X|} \mid 2^5 \cdot 3^2$. Thus $M \cong \mathbb{Z}_2^r$ or \mathbb{Z}_3^l , where $1 \leq r \leq 5$ and $1 \leq l \leq 2$. Let $L = MX$. Then $L = M : X$ and $L/C_L(M) \lesssim \text{Aut}(M) \cong \text{GL}(r, 2)$ or $\text{GL}(l, 3)$. Note that $M \leq C_L(M)$. If $M = C_L(M)$, then $L/C_L(M) = L/M \cong X \cong A_{80} \lesssim \text{GL}(r, 2)$ or $\text{GL}(l, 3)$. However, $\text{GL}(2, r)$ or $\text{GL}(3, l)$ has no subgroup isomorphic to A_{80} for $1 \leq r \leq 5$ and $1 \leq l \leq 2$. Hence we have $M < C_L(M)$ and $1 \neq C_L(M)/M \trianglelefteq L/M \cong A_{80}$. It follows that $A_{80} = C_L(M)/M$, that is, X centralizes M . Hence $L = M \times X$. Then $L_v/X_v \cong L/X \cong M$. It implies that L_v is soluble and $L_v \cong X_v.M$. Since $|X_v| = 80$, we have $|L_v| > 80$, a contradiction with Lemma 2.6. This completes the proof of the lemma. ■

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